Mesoscopic Modelling of Financial Markets

Stephane Cordier · Lorenzo Pareschi · Cyrille Piatecki

Abstract We derive a mesoscopic description of the behavior of a simple financial market where the agents can create their own portfolio between two investment alternatives: a stock and a bond. The model is derived starting from the Levy-Levy-Solomon microscopic model (Levy et al. in Econ. Lett. 45:103–111, 1994; Levy et al. in Microscopic Simulation of Financial Markets: From Investor Behavior to Market Phenomena, Academic Press, San Diego, 2000) using the methods of kinetic theory and consists of a linear Boltzmann equation for the wealth distribution of the agents coupled with an equation for the price of the stock. From this model, under a suitable scaling, we derive a Fokker-Planck equation and show that the equation admits a self-similar lognormal behavior. Several numerical examples are also reported to validate our analysis.

Keywords Wealth distribution · Power-law tails · Stock market · Self-similarity · Kinetic equations

1 Introduction

In recent years, physicists have been growing more and more interested in new interdisciplinary areas such as sociology and economics, originating what is today named socio-

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economical physics [1, 3, 4, 6, 9, 13, 15, 19, 20, 24, 27, 29]. This new area in physics borrows several methods and tools from classical statistical mechanics, where complex behavior arises from relatively simple rules due to the interaction of a large number of components. The motivation behind this is the attempt to identify and characterize universal and non-universal features in economical data in general.

A large part of the research in this area is concerned with power-law tails with universal exponents, as was predicted more than one century ago by Pareto [4, 14, 23]. In particular, by identifying the wealth in an economic system with the energy of a physical system, the application of statistical physics makes it possible to understand better the development of tails in wealth distributions. Starting from the microscopic dynamics, mesoscopic models can be derived with the tools of classical kinetic theory of fluids [1, 5–7, 13, 21, 22, 25].

In contrast with microscopic dynamics, where behavior often can be studied only empirically through computer simulations, kinetic models based on PDEs allow us to derive analytically general information on the model and its asymptotic behavior. For example, the knowledge of the large-wealth behavior is of primary importance, since it determines a posteriori whether the model can fit data of real economies.

In some recent papers, the explicit emergence of power laws in the wealth distribution, with Pareto index strictly larger than one, has been proved for open market economies where agents can interact through binary exchanges together with a simple source of speculative trading [1, 5, 22, 25].

The present work is motivated by the necessity to have a more realistic description of the speculative dynamics in the above models. To this end, we derive a mesoscopic description of the behavior of a simple financial market where a population of homogeneous agents can create their own portfolio between two investment alternatives: a stock and a bond. The model is closely related to the Levy-Levy-Solomon (LLS) microscopic model in finance [15, 16]. This model attempted to construct from simple rules complex behavior that could then mimic the market and explain the price formation mechanism. As a first step towards a more realistic description, we derive and analyze the model in the case of a single stock and under the assumption that the optimal proportion of investments is a function of the price only. In principle, several generalizations are possible (different stocks, heterogeneous agents, a time-dependent optimal proportion of investments, . . .), and we leave them for future investigations.

In our non-stationary financial market model, the average wealth is not conserved and this produces price variations. Let us point out that, even if the model is linear since no binary interaction dynamic between agents is present, the study of the large time behavior is not immediate. In fact, despite conservation of the total number of agents, we don't have any other additional conservation equation or entropy dissipation. Although we prove that the moments do not grow more than exponentially, the determination of an explicit form of the asymptotic wealth distribution of the kinetic equation remains difficult and requires the use of suitable numerical methods.

A complementary method to extract information on the tails is linked to the possibility to obtain particular asymptotics which maintain the characteristics of the solution to the original problem for large times. Following the analysis in [5], we shall prove that the Boltzmann model converges in a suitable asymptotic limit towards a convection-diffusion equation of Fokker-Planck type for the distribution of wealth among individuals. Other Fokker-Planck equations were obtained using different approaches in [1, 18, 26].

In this case, however, we can show that the Fokker-Planck equation admits self-similar solutions that can be computed explicitly and which are lognormal distributions. One is then led to the conclusion that the formation of Pareto tails in the wealth distribution observed in

[1, 5] is a consequence of the interplay between the conservative binary exchanges having the effect of redistributing wealth among agents and the speculative trading causing the growth of mean wealth and social inequalities.

The rest of the paper is organized as follows. In the next section, we introduce briefly the microscopic dynamic of the LLS model. The mesoscopic model is then derived in Sect. 3 and its properties discussed in Sect. 4. These properties justify the asymptotic procedures performed in Sect. 5. The model behavior together with its asymptotic limit is illustrated by several numerical results in Sect. 6. Some conclusions and remarks on future developments are then made in the last section.

2 The Microscopic Dynamic

Let us consider a set of financial agents $i = 1, \ldots, N$ who can create their own portfolio between two alternative investments: a stock and a bond. We denote by w_i the wealth of agent i and by n_i the number of stocks of the agent. Additionally we use the notations S for the price of the stock and n for the total number of stocks.

The essence of the dynamic is the choice of the agent's portfolio. More precisely, at each time step each agent selects which fraction of wealth to invest in bonds and which fraction in stocks. We indicate with r the (constant) interest rate of bonds. The bond is assumed to be a risk-less asset yielding a return at the end of each time period. The stock is a risky asset with overall returns rate x composed of two elements: a capital gain or loss and the distribution of dividends.

To simplify the notation, let us neglect for the moment the effects due to the stochastic nature of the process, the presence of dividends, and so on. Thus, if an agent has invested $\gamma_i w_i$ of its wealth in stocks and $(1 - \gamma_i)w_i$ of its wealth in bonds, at the next time step in the dynamic he will achieve the new wealth value

$$w_i' = (1 - \gamma_i)w_i(1 + r) + \gamma_i w_i(1 + x), \tag{1}$$

where the rate of return of the stock is given by

$$x = \frac{S' - S}{S}, \tag{2}$$

and S' is the new price of the stock.

Since we have the identity

$$\gamma_i w_i = n_i S, \tag{3}$$

we can also write

$$w_i' = w_i + w_i(1 - \gamma_i)r + w_i \gamma_i \left(\frac{S' - S}{S} \right) \tag{4}$$

$$= w_i + (w_i - n_i S)r + n_i(S' - S). \tag{5}$$

Note that, independently of the number of stocks of the agent at the next time level, it is only the price variation of the stock (which is unknown) that characterizes the gain or loss of the agent on the stock market at this stage.

The dynamic now is based on the agent choice of the new fraction of wealth he wants to invest in stocks at the next stage. Each investor i is confronted with a decision where

the outcome is uncertain: which is the new optimal fraction γ_i' of wealth to invest in stock? According to the standard theory of investment each investor is characterized by a *utility function* (of its wealth) $U(w)$ that reflects the personal risk taking preference [12]. The optimal γ_i' is the one that maximizes the expected value of $U(w)$.

Different models can be used for this (see [16, 29]), for example, maximizing a von Neumann-Morgenstern utility function with a constant risk aversion of the type

$$U(w) = \frac{w^{1-\alpha}}{1-\alpha}, \tag{6}$$

where α is the risk aversion parameter, or a logarithmic utility function

$$U(w) = \log(w). \tag{7}$$

As they don't know the future stock price S', the investors estimate the stock's next period return distribution and find an optimal mix of the stock and the bond that maximizes their expected utility $E[U]$. In practice, for any hypothetical price S^h, each investor finds the hypothetical optimal proportion $\gamma_i^h(S^h)$ which maximizes his/her expected utility evaluated at

$$w_i^h(S^h) = (1 - \gamma_i^h)w_i'(1 + r) + \gamma_i^h w_i'(1 + x'(S^h)), \tag{8}$$

where $x'(S^h) = (S^h - S')/S'$ and S' is estimated in some way. For example in [16] the investors expectations for x' are based on extrapolating the past values.

Note that, if we assume that all investors have the same risk aversion α in (6), then they will have the same proportion of investment in stocks regardless of their wealth, thus $\gamma_i^h(S^h) = \gamma^h(S^h)$.

Once each investor decides on the hypothetical optimal proportion of wealth γ_i^h that he/she wishes to invest in stocks, one can derive the number of stocks $n_i^h(S^h)$ he/she wishes to hold corresponding to each hypothetical stock price S^h. Since the total number of shares in the market n, is fixed there is a particular value of the price S' for which the sum of the $n_i^h(S^h)$ equals n. This value S' is the new market equilibrium price and the optimal proportion of wealth is $\gamma_i' = \gamma_i^h(S')$.

More precisely, following [16], each agent formulates a *demand curve*

$$n_i^h = n_i^h(S^h) = \frac{\gamma^h(S^h)w_i^h(S^h)}{S^h}$$

characterizing the desired number of stocks as a function of the hypothetical stock price S^h. This number of share demands is a monotonically decreasing function of the hypothetical price S^h. As the total number of stocks

$$n = \sum_{i=1}^{N} n_i \tag{9}$$

is preserved, the new price of the stock at the next time level is given by the so-called *market clearance condition*. Thus the new stock price S' is the unique price at which the total demand equals the supply

$$\sum_{i=1}^{N} n_i^h(S') = n. \tag{10}$$

This will fix the value w' in (1) and the model can be advanced to the next time level. To make the model more realistic, typically a source of stochastic noise, which characterizes all factors causing the investor to deviate from his/her optimal portfolio, is introduced in the proportion of investments γ_i and in the rate of return of the stock x'.

3 Kinetic Modelling

We define $f = f(w, t)$, $w \in \mathbb{R}_+$, $t > 0$ the distribution of wealth w, which represents the probability for an agent to have a wealth w. We assume that at time t the percentage of wealth invested is of the form $\gamma(\xi) = \mu(S) + \xi$, where ξ is a random variable in $[-z, z]$, and $z = \min\{-\mu(S), 1 - \mu(S)\}$ is distributed according to some probability density $\Phi(\mu(S), \xi)$ with zero mean and variance ζ^2. This probability density characterizes the individual strategy of an agent around the optimal choice $\mu(S)$. We assume Φ to be independent of the wealth of the agent. Here, the optimal demand curve $\mu(\cdot)$ is assumed to be a given monotonically non-increasing function of the price $S \geq 0$ such that $0 < \mu(0) < 1$.

Note that given $f(w, t)$ the actual stock price S satisfies the demand-supply relation

$$S = \frac{1}{n} E[\gamma w], \tag{11}$$

where $E[X]$ denotes the mathematical expectation of the random variable X and $f(w, t)$ has been normalized:

$$\int_0^\infty f(w, t)dw = 1.$$

More precisely, since γ and w are independent, at each time t, the price $S(t)$ satisfies (see Fig. 1)

$$S(t) = \frac{1}{n} E[\gamma] E[w] = \frac{1}{n} \mu(S(t)) \bar{w}(t), \tag{12}$$

with

$$\bar{w}(t) \overset{def}{:=} E[w] = \int_0^\infty f(w, t)w dw \tag{13}$$

being the mean wealth and by construction,

$$\mu(S) = \int \Phi(\mu(S), \xi) \xi \, d\xi.$$

At the next round in the market, the new wealth of the investor will depend on the future price S' and the percentage γ of wealth invested according to
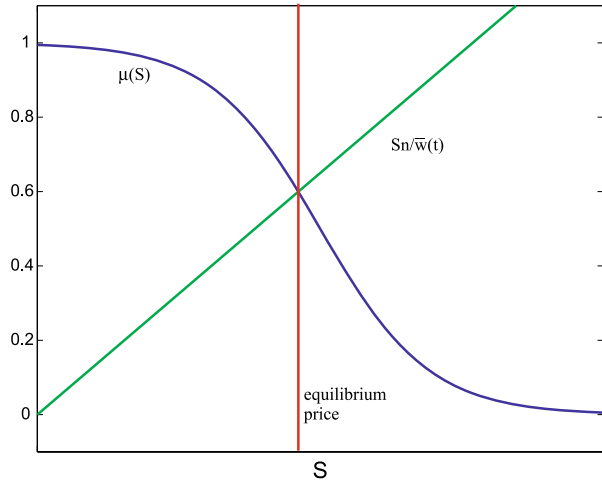
$$w'(S', \gamma, \eta) = (1 - \gamma)w(1 + r) + \gamma w(1 + x(S', \eta)), \tag{14}$$

where the expected rate of return of stocks is given by

$$x(S', \eta) = \frac{S' - S + D + \eta}{S}. \tag{15}$$

In the above relation, $D \geq 0$ represents a constant dividend paid by the company and η is a random variable distributed according to $\Theta(\eta)$ with zero mean and variance σ^2, which takes

into account fluctuations due to price uncertainty and dividends [11, 16]. We assume η to take values in $[-d, d]$ with $0 < d \le S' + D$ so that $w' \ge 0$ and thus negative wealths are not allowed in the model. Note that (15) requires estimation of the future price S', which is unknown.

The dynamic is then determined by the agent's new fraction of wealth invested in stocks, $\gamma'(\xi') = \mu(S') + \xi'$, where ξ' is a random variable in $[-z', z']$ and $z' = \min\{\mu(S'), 1 - \mu(S')\}$ is distributed according to $\Phi(\mu(S'), \xi')$. We have the demand-supply relation

$$S' = \frac{1}{n} E[\gamma' w'], \tag{16}$$

which permits us to write the following equation for the future price:

$$S' = \frac{1}{n} E[\gamma'] E[w'] = \frac{1}{n} \mu(S') E[w']. \tag{17}$$

Now

$$w'(S', \gamma, \eta) = w(1 + r) + \gamma w(x(S', \eta) - r), \tag{18}$$

thus

$$E[w'] = E[w](1 + r) + E[\gamma w](E[x(S', \eta)] - r) \tag{19}$$

$$= \bar{w}(t)(1 + r) + \mu(S)\bar{w}(t) \left(\frac{S' - S + D}{S} - r \right). \tag{20}$$

This gives the identity

$$S' = \frac{1}{n} \mu(S') \bar{w}(t) \left[(1 + r) + \mu(S) \left(\frac{S' - S + D}{S} - r \right) \right]. \tag{21}$$

Using (12) we can eliminate the dependence on the mean wealth and write

$$S' = \frac{\mu(S')}{\mu(S)} \left[(1 - \mu(S))S(1+r) + \mu(S)(S' + D) \right]$$

$$= \frac{(1 - \mu(S))\mu(S')}{(1 - \mu(S'))\mu(S)}(1+r)S + \frac{\mu(S')}{1 - \mu(S')}D. \tag{22}$$

Remark 3.1 The equation for the future price deserves some remarks.

• Equation (22) determines implicitly the future value of the stock price. Let us set

$$g(S) = \frac{1 - \mu(S)}{\mu(S)}S.$$

Then the future price is given by the equation

$$g(S') = g(S)(1+r) + D$$

for a given S. Note that

$$\frac{dg(S)}{dS} = -\frac{d\mu(S)}{dS}\frac{S}{\mu(S)^2} + \frac{1 - \mu(S)}{\mu(S)} > 0,$$

so the function $g(S)$ is strictly increasing with respect to S. This guarantees the existence of a unique solution

$$S' = g^{-1}(g(S)(1+r) + D) > S. \tag{23}$$

Moreover, if $r = 0$ and $D = 0$, the unique solution is $S' = S$ and the price remains unchanged in time.

For the average stock return, we have

$$\bar{x}(S') - r = \frac{(\mu(S') - \mu(S))(1+r)}{(1 - \mu(S'))\mu(S)} + \frac{\mu(S')D}{S(1 - \mu(S'))}, \tag{24}$$

where

$$\bar{x}(S') = E[x(S', \eta)] = \frac{S' - S + D}{S}. \tag{25}$$

Now the right hand side of (24) has non-constant sign since $\mu(S') \leq \mu(S)$. In particular, the average stock return is above the bonds rate r only if the (negative) rate of variation of the investments is above a certain threshold

$$\frac{\mu(S') - \mu(S)}{\mu(S)\mu(S')}S \geq -\frac{D}{(1+r)}.$$

• In the constant investment case $\mu(\cdot) = C$, with $C \in (0, 1)$ constant, then we have $g(S) = (1 - C)S/C$ and

$$S' = (1+r)S + \frac{C}{1-C}D,$$

which corresponds to a dynamic of growth of the prices at rate r. As a consequence, the average stock return is always larger then the constant return of bonds:

$$\bar{x}(S') - r = \frac{D}{S(1-C)} \geq 0.$$

By standard methods of kinetic theory [2], the microscopic dynamics of agents originate the following linear kinetic equation for the evolution of the wealth distribution:

$$\frac{\partial f(w,t)}{\partial t} = \int_{-d}^{d} \int_{-z}^{z} \left(\beta('w \to w) \frac{1}{j(\xi, \eta, t)} f('w, t) - \beta(w \to w') f(w, t) \right) d\xi \, d\eta. \quad (26)$$

The above equation takes into account all possible variations that can occur to the distribution of a given wealth w. The first part of the integral on the right hand side takes into account all possible gains of the test wealth w coming from a pre-trading wealth $'w$. The function $\beta('w \to w)$ gives the probability per unit time of this process.

Thus $'w$ is obtained simply by inverting the dynamics to get

$$'w = \frac{w}{j(\xi, \eta, t)}, \quad j(\xi, \eta, t) = 1 + r + \gamma(\xi)(x(S', \eta) - r), \quad (27)$$

where the value S' is given as the unique fixed point of (17).

The presence of the term j in the integral is needed in order to preserve the total number of agents

$$\frac{d}{dt} \int_{0}^{\infty} f(w,t) dw = 0.$$

The second part of the integral on the right hand side of (26) is a negative term that takes into account all possible losses of wealth w as a consequence of the direct dynamic (14), the rate of this process now being $\beta(w \to w')$. In our case, the kernel β takes the form

$$\beta(w \to w') = \Phi(\mu(S), \xi)\Theta(\eta). \quad (28)$$

The distribution function $\Phi(\mu(S), \xi)$, together with the function $\mu(\cdot)$, characterizes the behavior of the agents on the market (more precisely, they characterize the way the agents invest their wealth as a function of the actual price of the stock).

Remark 3.2 In the derivation of the kinetic equation, we assumed for simplicity that the actual demand curve $\mu(\cdot)$ which gives the optimal proportion of investments is a function of the price only. In reality, the demand curve should change at each market iteration and should thus depend also on time. In the general case where each agent has a wealth-dependent individual strategy, one should consider the distribution $f(\gamma, w, t)$ of agents having a fraction γ of their wealth w invested in stocks.

4 Properties of the Kinetic Equation

We will start our analysis by introducing some notations. Let $\mathcal{M}_0$ be the space of all probability measures on $\mathbb{R}_+$ and by

$$\mathcal{M}_p = \left\{ \Psi \in \mathcal{M}_0 : \int_{\mathbb{R}_+} |\vartheta|^p \Psi(\vartheta) \, d\vartheta < +\infty, \ p \geq 0 \right\}, \quad (29)$$

we mean the space of all Borel probability measures of finite momentum of order p, equipped with the topology of the weak convergence of the measures.

Let $\mathcal{F}_p(\mathbb{R}_+)$, $p > 1$ be the class of all real functions on $\mathbb{R}_+$ such that $g(0) = g'(0) = 0$ and $g^{(m)}(v)$ is Hölder continuous of order δ,

$$\|g^{(m)}\|_\delta = \sup_{v \neq w} \frac{|g^{(m)}(v) - g^{(m)}(w)|}{|v - w|^\delta} < \infty, \tag{30}$$

the integer m and the number $0 < \delta \leq 1$ be such that $m + \delta = p$, and $g^{(m)}$ denote the m-th derivative of g.

Clearly the symmetric probability density Θ which characterizes the stock returns belongs to $\mathcal{M}_p$ for all $p > 0$ since

$$\int_{-d}^{d} |\eta|^p \Theta(\eta) d\eta \leq |d|^p.$$

Moreover, to simplify computations, we assume that this density is obtained from a given random variable Y with zero mean and unit variance. Thus Θ of variance σ^2 is the density of σY. By this assumption, we can easily obtain the dependence on σ of the moments of Θ. In fact, for any $p > 2$,

$$\int_{-d}^{d} |\eta|^p \Theta(\eta) d\eta = E\left(|\sigma Y|^p\right) = \sigma^p E\left(|Y|^p\right).$$

Note that (26) in weak form takes the simpler form

$$\frac{d}{dt} \int_0^\infty f(w,t)\phi(w)dw$$

$$= \int_0^\infty \int_{-D}^{D} \int_{-z}^{z} \Phi(\mu(S),\xi)\Theta(\eta) f(w,t)(\phi(w') - \phi(w))d\xi\, d\eta\, dw. \tag{31}$$

By a weak solution of the initial value problem for (26) corresponding to the initial probability density $f_0(w) \in \mathcal{M}_p$, $p > 1$, we shall mean any probability density $f \in C^1(\mathbb{R}_+, \mathcal{M}_p)$ satisfying the weak form (31) for $t > 0$ and all $\phi \in \mathcal{F}_p(\mathbb{R}_+)$, and such that for all $\phi \in \mathcal{F}_p(\mathbb{R}_+)$,

$$\lim_{t \to 0} \int_0^\infty f(w,t)\phi(w)\, dw = \int_0^\infty f_0(w)\phi(w)\, dw. \tag{32}$$

The form (31) is easier to handle, and it is the starting point to study the evolution of macroscopic quantities (moments). The existence of a weak solution to (26) can be seen easily using the same methods available for the linear Boltzmann equation (see [28] and the references therein for example).

From (31) follows the conservation of the total number of investors if $\phi(w) = 1$. The choice $\phi(w) = w$ is of particular interest since it gives the time evolution of the average wealth which characterizes the price behavior. In fact, the mean wealth is not conserved in the model since we have

$$\frac{d}{dt} \int_0^\infty f(w,t)w\, dw = \left(r + \mu(S)\left(\frac{S' - S + D}{S} - r\right)\right) \int_0^\infty f(w,t)w\, dw. \tag{33}$$

Note that since the sign of the right hand side is nonnegative, the mean wealth is nondecreasing in time. In particular, we can rewrite the equation as

$$\frac{d}{dt}\bar{w}(t) = \left((1 - \mu(S))r + \mu(S)\bar{x}(S')\right)\bar{w}(t). \tag{34}$$

From this we get the equation for the price

$$\frac{d}{dt}S(t) = \frac{\mu(S(t))}{\mu(S(t)) - \dot{\mu}(S(t))S(t)}\left((1 - \mu(S(t)))r + \mu(S(t))\bar{x}(S'(t))\right)S(t), \tag{35}$$

where S' is given by (22) and

$$\dot{\mu}(S) = \frac{d\mu(S)}{dS} \leq 0.$$

Now since from (24) it follows by the monotonicity of μ that

$$\bar{x}(S') \leq M \overset{def}{:=} r + \frac{D}{S(0)(1 - \mu(S(0)))},$$

using (34) we have the bound

$$\bar{w}(t) \leq \bar{w}(0)\exp(Mt). \tag{36}$$

From (12) we obtain immediately

$$\frac{S(t)}{\mu(S(t))} \leq \frac{S(0)}{\mu(S(0))}\exp(Mt),$$

which gives

$$S(t) \leq S(0)\exp(Mt). \tag{37}$$

Remark 4.1 For a constant $\mu(\cdot) = C$, $C \in (0, 1)$ we have the explicit expression for the growth of the wealth (and consequently of the price)

$$\bar{w}(t) = \bar{w}(0)\exp(rt) - (1 - \exp(rt))\frac{nD}{1 - C}. \tag{38}$$

Analogous bounds to (36) for moments of higher order can be obtained in a similar way. Let us consider the case of moments of order $p \geq 2$, which we will need in the sequel. Taking $\phi(w) = w^p$, we get

$$\frac{d}{dt}\int_0^\infty w^p f(w, t)\,dw = \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi)\Theta(\eta)f(w, t)(w'^p - w^p)d\xi\,d\eta\,dw. \tag{39}$$

Moreover, we can write

$$w'^p = w^p + pw^{p-1}(w' - w) + \frac{1}{2}p(p - 1)\tilde{w}^{p-2}(w' - w)^2,$$

where, for some $0 \leq \vartheta \leq 1$,

$$\tilde{w} = \vartheta w' + (1 - \vartheta)w.$$

Hence,

$$\int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S),\xi)\Theta(\eta)f(w,t)(w'^p - w^p)d\xi\,d\eta\,dw$$

$$= \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S),\xi)\Theta(\eta)f(w,t)(pw^{p-1}(w'-w)$$

$$+ \frac{1}{2}p(p-1)\tilde{w}^{p-2}(w'-w)^2)d\xi\,d\eta\,dw$$

$$= p((1-\mu(S))r + \mu(S)\bar{x}(S'))\int_0^\infty w^p f(w,t)\,dw + \frac{1}{2}p(p-1)$$

$$\times \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S),\xi)\Theta(\eta)f(w,t)\tilde{w}^{p-2}w^2((1-\gamma)r+\gamma x(S',\eta))^2d\xi\,d\eta\,dw.$$

From

$$\tilde{w}^{p-2} = w^{p-2}(1+\vartheta((1-\gamma)r+\gamma x(S',\eta)))^{p-2} \le w^{p-2}(1+r+|x(S',\eta)|)^{p-2}$$

$$\le C_p w^{p-2}(1+r^{p-2}+|x(S',\eta)|^{p-2})$$

and

$$((1-\gamma)r+\gamma x(S',\eta)))^2 \le 2(r^2+x(S',\eta)^2),$$

we have

$$\int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S),\xi)\Theta(\eta)f(w,t)\tilde{w}^{p-2}w^2((1-\gamma)r+\gamma x(S',\eta))^2d\xi\,d\eta\,dw$$

$$\le 2C_p \int_0^\infty \int_{-d}^d \Theta(\eta)f(w,t)w^p(1+r^{p-2}+|x(S',\eta)|^{p-2})(r^2+x(S',\eta)^2)d\eta\,dw.$$

Since

$$\int_{-d}^d \Theta(\eta)|x(S',\eta)|^p\,d\eta \le \frac{c_p}{S^p}\left((S'-S)^p + D^p + \sigma^p E(|Y|^p)\right), \tag{40}$$

we finally obtain the bound

$$\frac{d}{dt}\int_0^\infty w^p f(w,t)\,dw \le A_p(S)\int_0^\infty w^p f(w,t)\,dw, \tag{41}$$

where

$$A_p(S) = p((1-\mu(S))r + \mu(S)\bar{x}(S'))$$

$$+ p(p-1)C_p\Bigg[r^p + (1+r^{p-2})\left(1+\frac{c_2}{S_2}((S'-S)^2+D^2+\sigma^2 E(|Y|^2))\right)$$

$$+ r^2\left(1+\frac{c_{p-2}}{S^{p-2}}((S'-S)^{p-2}+D^{p-2}+\sigma^{p-2}E(|Y|^{p-2}))\right)$$

$$+ \left(\frac{c_p}{S^p}((S'-S)^p + D^p + \sigma^p E(|Y|^p))\right)\Bigg]$$

and C_p, c_p, c_{p-2} and c_2 are suitable constants.

We can summarize our results in the following

Theorem 4.1 *Let the probability density $f_0 \in \mathcal{M}_p$, where $p = 2 + \delta$ for some $\delta > 0$. Then the average wealth is increasing exponentially with time following* (36). *As a consequence, if μ is a non-increasing function of S, the price does not grow more than exponentially as in* (37). *Similarly, higher order moments do not increase more than exponentially, and we have the bound* (41).

5 Fokker-Planck Asymptotics and Self-Similar Solution

The previous analysis shows that in general it is difficult to study in detail the asymptotic behavior of the system. In addition, we must take into account the exponential growth of the average wealth. In this case, one way to get information on the properties of the solution for large time relies on a suitable scaling of the solution. As is usual in kinetic theory, however, particular asymptotics of the equation result in simplified models (generally of Fokker-Planck type) whose behavior is easier to analyze. Here, following the analysis in [5, 22] and inspired by similar asymptotic limits for inelastic gases [8, 25], we consider the limit of large times in which the market originates a very small exchange of wealth (small rates of return r and x).

In order to study the asymptotic behavior of the distribution function $f(w, t)$, we start from the weak form of the kinetic equation

$$\frac{d}{dt} \int_0^\infty f(w, t)\phi(w)dw$$

$$= \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi)\Theta(\eta) f(w, t)(\phi(w') - \phi(w))d\xi \, d\eta \, dw \qquad (42)$$

and consider a second-order Taylor expansion of ϕ around w,

$$\phi(w') - \phi(w) = w(r + \gamma(x(S', \eta) - r))\phi'(w) + \frac{1}{2}w^2(r + \gamma(x(S', \eta) - r))^2\phi''(\tilde{w}),$$

where, for some $0 \leq \vartheta \leq 1$,

$$\tilde{w} = \vartheta w' + (1 - \vartheta)w.$$

Inserting this expansion into the collision operator, we get

$$\int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi)\Theta(\eta) f(w, t)(\phi(w') - \phi(w))d\xi \, d\eta \, dw$$

$$= \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi)\Theta(\eta) f(w, t)w(r + \gamma(x(S', \eta) - r))\phi'(w)d\xi \, d\eta \, dw$$

$$+ \frac{1}{2} \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi)\Theta(\eta) f(w, t)w^2(r + \gamma(x(S', \eta) - r))^2\phi''(w)d\xi \, d\eta \, dw$$

$$+ R_r(S, S'),$$

where

$$R_r(S, S') = \frac{1}{2} \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi)\Theta(\eta) f(w, t)$$
$$\times w^2 (r + \gamma(x(S', \eta) - r))^2 (\phi''(\tilde{w}) - \phi''(w)) \, d\xi \, d\eta \, dw. \qquad (43)$$

Recalling that $E[\xi] = 0$, $E[\eta] = 0$, $E[\xi^2] = \zeta^2$ and $E[\eta^2] = \sigma^2$, we can simplify the above expression to obtain

$$\int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S), \xi)\Theta(\eta) f(w, t)(\phi(w') - \phi(w)) d\xi \, d\eta \, dw$$
$$= \int_0^\infty f(w, t) w \left(r + \mu(S) \left(\frac{S' - S + D}{S} - r \right) \right) \phi'(w) \, dw$$
$$+ \frac{1}{2} \int_0^\infty f(w, t) w^2 \left(r^2 + (\zeta^2 + \mu(S)^2) \left(\frac{(S' - S)^2}{S^2} + \frac{\sigma^2 + D^2}{S^2} + 2D \frac{S' - S}{S^2} \right. \right.$$
$$+ r^2 - 2r \frac{S' - S + D}{S} \bigg) + 2r\mu(S) \left(\frac{S' - S + D}{S} - r \right) \bigg) \phi''(w) \, dw$$
$$+ R_r(S, S').$$

Now we set

$$\tau = rt, \qquad \tilde{f}(w, \tau) = f(w, t), \qquad \tilde{S}(\tau) = S(t), \qquad \tilde{\mu}(\tilde{S}) = \mu(S),$$

which implies that $\tilde{f}(w, \tau)$ satisfies the equation

$$\frac{d}{d\tau} \int_0^\infty \tilde{f}(w, \tau) \phi(w) dw$$
$$= \int_0^\infty \tilde{f}(w, \tau) w \left(1 + \tilde{\mu}(\tilde{S}) \left(\frac{\tilde{S}' + D - \tilde{S}}{r\tilde{S}} - 1 \right) \right) \phi'(w) \, dw$$
$$+ \frac{1}{2} \int_0^\infty \tilde{f}(w, \tau) w^2 \left(r + (\zeta^2 + \tilde{\mu}(\tilde{S})^2) \left(\frac{(\tilde{S}' - \tilde{S})^2}{r\tilde{S}^2} + \frac{\sigma^2 + D^2}{r\tilde{S}^2} + 2D \frac{\tilde{S}' - \tilde{S}}{r\tilde{S}^2} \right. \right.$$
$$+ r - 2 \frac{\tilde{S}' + D - \tilde{S}}{\tilde{S}} \bigg) + 2\tilde{\mu}(\tilde{S}) \left(\frac{\tilde{S}' + D - \tilde{S}}{\tilde{S}} - r \right) \bigg) \phi''(w) \, dw$$
$$+ \frac{1}{r} R_r(\tilde{S}, \tilde{S}').$$

Now we consider the limit of very small values of the constant rate r. In order for such a limit to make sense and preserve the characteristics of the model, we must assume that

$$\lim_{r \to 0} \frac{\sigma^2}{r} = \nu, \qquad \lim_{r \to 0} \frac{D}{r} = \lambda. \qquad (44)$$

First let us note that the above limits in (22) imply immediately that

$$\lim_{r \to 0} \tilde{S}' = \tilde{S}. \qquad (45)$$

We begin by showing that the remainder is small for small values of r.

Since $\phi \in \mathcal{F}_{2+\delta}(\mathbb{R}_+)$ and $|\tilde{w} - w| = \vartheta |w' - w|$,

$$\left| \phi''(\tilde{w}) - \phi''(w) \right| \leq \|\phi''\|_\delta |\tilde{w} - w|^\delta \leq \|\phi''\|_\delta |w' - w|^\delta. \tag{46}$$

Hence

$$\left| \frac{1}{r} R_r(\tilde{S}, \tilde{S}') \right| \leq \frac{\|\phi''\|_\delta}{2r} \int_0^\infty \int_{-d}^d \int_{-z}^z \Phi(\mu(S(\tau)), \xi) \Theta(\eta)$$
$$\times |(1-\gamma)r + \gamma x(\tilde{S}', \eta)|^{2+\delta} \tilde{f}(w, \tau) w^{2+\delta} d\xi \, d\eta \, dw.$$

By the inequality

$$|(1-\gamma)r + \gamma x(\tilde{S}', \eta)|^{2+\delta} \leq 2^{2+\delta} \left(r^{2+\delta} + |x(\tilde{S}', \eta)|^{2+\delta} \right)$$

and (40), we get

$$\left| \frac{1}{r} R_r(\tilde{S}, \tilde{S}') \right| \leq 2^{1+\delta} \|\phi''\|_\delta \left(r^{1+\delta} + \frac{c_{2+\delta}}{r \tilde{S}^{2+\delta}} ((\tilde{S}' - \tilde{S})^{2+\delta} + D^{2+\delta} + \sigma^{2+\delta} E(|Y|^{2+\delta})) \right)$$
$$\times \int_0^\infty \tilde{f}(w, \tau) w^{2+\delta} dw.$$

As a consequence of (44)–(45), from this inequality it follows that $R_r(\tilde{S}, \tilde{S}')$ converges to zero as $r \to 0$ if

$$\int_0^\infty w^{2+\delta} \tilde{f}(w, \tau) \, dw$$

remains bounded at any fixed time $\tau > 0$, provided the same bound holds at time $\tau = 0$. This is guaranteed by inequality (41) since $A_p(\tilde{S}) \to 0$ in the asymptotic limit defined by (44).

Next we write

$$\tilde{\mu}(\tilde{S}') = \tilde{\mu}(\tilde{S}) + (\tilde{S}' - \tilde{S})\dot{\tilde{\mu}}(\tilde{S}) + O((\tilde{S}' - \tilde{S})^2),$$

where

$$\dot{\tilde{\mu}}(\tilde{S}) = \frac{d\tilde{\mu}(\tilde{S})}{d\tilde{S}} \leq 0.$$

Then, using the above expansion from (44) in (22), we obtain

$$\lim_{r \to 0} \frac{\tilde{S}' - \tilde{S}}{r} = \kappa(\tilde{S}) \left(\tilde{S} + \frac{\tilde{\mu}(\tilde{S})}{1 - \tilde{\mu}(\tilde{S})} \lambda \right), \tag{47}$$

with

$$0 < \kappa(\tilde{S}) \overset{def}{:=} \frac{\tilde{\mu}(\tilde{S})(1 - \tilde{\mu}(\tilde{S}))}{\tilde{\mu}(\tilde{S})(1 - \tilde{\mu}(\tilde{S})) - \tilde{S}\dot{\tilde{\mu}}(\tilde{S})} \leq 1. \tag{48}$$

Now, sending $r \to 0$ under the same assumptions, we obtain the weak form

$$\frac{d}{d\tau} \int_0^\infty \tilde{f}(w, \tau) \phi(w) dw$$

$$= \left(1 + \tilde{\mu}(\tilde{S}) \left((\kappa(\tilde{S}) - 1) + \frac{\tilde{\mu}(\tilde{S})(\kappa(\tilde{S}) - 1) + 1}{1 - \tilde{\mu}(\tilde{S})} \frac{\lambda}{\tilde{S}} \right) \right) \int_0^\infty \tilde{f}(w, \tau) w \phi'(w) \, dw$$

$$+ \frac{1}{2} \frac{(\tilde{\mu}(\tilde{S})^2 + \zeta^2)}{\tilde{S}^2} \nu \int_0^\infty \tilde{f}(w, \tau) w^2 \phi''(w) \, dw.$$

This corresponds to the weak form of the Fokker-Planck equation

$$\frac{\partial}{\partial \tau} \tilde{f} + A(\tau) \frac{\partial}{\partial w} (w \tilde{f}) = \frac{1}{2} B(\tau) \frac{\partial^2}{\partial w^2} (w^2 \tilde{f}),$$

or equivalently

$$\frac{\partial}{\partial \tau} \tilde{f} = \frac{\partial}{\partial w} \left[-A(\tau) w \tilde{f} + \frac{1}{2} B(\tau) \frac{\partial}{\partial w} w^2 \tilde{f} \right], \tag{49}$$

with

$$A(\tau) = 1 + \tilde{\mu}(\tilde{S}) \left((\kappa(\tilde{S}) - 1) + \frac{\tilde{\mu}(\tilde{S})(\kappa(\tilde{S}) - 1) + 1}{1 - \tilde{\mu}(\tilde{S})} \frac{\lambda}{\tilde{S}} \right), \tag{50}$$

$$B(\tau) = \frac{(\tilde{\mu}(\tilde{S})^2 + \zeta^2)}{\tilde{S}^2} \nu. \tag{51}$$

Thus we have proved

Theorem 5.1 *Let the probability density $f_0 \in \mathcal{M}_p$, where $p = 2 + \delta$ for some $\delta > 0$. Then, as $r \to 0$, $\sigma \to 0$, and $D \to 0$ in such a way that $\sigma^2 = \nu r$ and $D = \lambda r$, the weak solution to the Boltzmann equation* (31) *for the scaled density $\tilde{f}_r(w, \tau) = f(v, t)$ with $\tau = rt$ converges, up to extraction of a subsequence, to a probability density $\tilde{f}(w, \tau)$. This density is a weak solution of the Fokker-Planck equation* (49).

We remark that even for the Fokker-Planck model, the mean wealth is increasing with time. A simple computation shows that

$$\dot{\bar{w}}(\tau) = \frac{d}{d\tau} \int_0^\infty \tilde{f}(w, \tau) w \, dw = A(\tau) \int_0^\infty \tilde{f}(w, \tau) w \, dw = A(\tau) \bar{w}(\tau). \tag{52}$$

Using (48), we get the bound

$$(1 - \tilde{\mu}(\tilde{S})) \bar{w}(\tau) + n\lambda \leq \dot{\bar{w}}(\tau) \leq \bar{w}(\tau) + \frac{n\lambda}{1 - \tilde{\mu}(\tilde{S})}. \tag{53}$$

Similarly, for the second-order moment we have

$$\dot{\bar{e}}(\tau) = \frac{d}{d\tau} \int_0^\infty \tilde{f}(w, \tau) w^2 \, dw = (2A(\tau) + B(\tau)) \int_0^\infty \tilde{f}(w, \tau) w^2 \, dw$$

$$= (2A(\tau) + B(\tau)) \bar{e}(\tau). \tag{54}$$

In order to search for self-similar solutions, we consider the scaling

$$\tilde{f}(w, \tau) = \frac{1}{w} \tilde{g}(\chi, \tau), \quad \chi = \log(w).$$

Simple computations show that $\tilde{g}(\chi, \tau)$ satisfies the linear convection-diffusion equation

$$\frac{\partial}{\partial \tau} \tilde{g}(\chi, \tau) = \left(\frac{B(\tau)}{2} - A(\tau) \right) \frac{\partial}{\partial \chi} \tilde{g}(\chi, \tau) + \frac{B(\tau)}{2} \frac{\partial^2}{\partial \chi^2} \tilde{g}(\chi, \tau),$$

which admits the self-similar solution (see [17] for example)

$$\tilde{g}(\chi, \tau) = \frac{1}{(2b(\tau)\pi)^{1/2}} \exp\left(-\frac{(\chi + b(\tau)/2 - a(\tau))^2}{2b(\tau)} \right), \tag{55}$$

where

$$a(\tau) = \int_0^\tau A(s)\, ds + C_1, \qquad b(\tau) = \int_0^\tau B(s)\, ds + C_2.$$

Reverting to the original variables, we obtain the lognormal asymptotic behavior of the model,

$$\tilde{f}(w, \tau) = \frac{1}{w(2b(\tau)\pi)^{1/2}} \exp\left(-\frac{(\log(w) + b(\tau)/2 - a(\tau))^2}{2b(\tau)} \right). \tag{56}$$

The constants $C_1 = a(0)$ and $C_2 = b(0)$ can be determined from the initial data at $t = 0$. If we denote by $\bar{w}(0)$ and $\bar{e}(0)$ the initial values of the first two central moments, we get

$$C_1 = \log(\bar{w}(0)), \qquad C_2 = \log\left(\frac{\bar{e}(0)}{(\bar{w}(0))^2} \right).$$

Finally, a direct computation shows that

$$a(\tau) = \int_0^\tau \frac{\dot{\bar{w}}(s)}{\bar{w}(s)}\, ds + C_1 = \log(\bar{w}(\tau)) \tag{57}$$

and

$$b(\tau) = \int_0^\tau \left(\frac{\dot{\bar{e}}(s)}{\bar{e}(s)} - 2\frac{\dot{\bar{w}}(s)}{\bar{w}(s)} \right) ds + C_2 = \log\left(\frac{\bar{e}(\tau)}{(\bar{w}(\tau))^2} \right). \tag{58}$$

Remark 5.1

- If we assume ζ and σ are of the same order of magnitude, in the Fokker-Planck limit the noise introduced by the agents' deviations with respect to their optimal behavior does not play any role and the only source of diffusion is due to the stochastic nature of the returns.
- In the case of constant investments $\tilde{\mu}(\cdot) = C$, $C \in (0, 1)$ we have the simplified Fokker-Planck equation

$$\frac{\partial}{\partial \tau} \tilde{f} = \frac{\partial}{\partial w} \left[-\left(1 + \frac{C}{1-C} \frac{\lambda}{\bar{S}} \right) w\tilde{f} + \frac{1}{2} \frac{(C^2 + \zeta^2)}{\bar{S}^2} v \frac{\partial}{\partial w} w^2 \tilde{f} \right].$$

It is easy to verify that for such a simple situation, the pair of ordinary differential equations for the evolution of the first two central moments, (52) and (54), can be solved explicitly.

- Imposing the conservation of the mean wealth with the scaling

$$\tilde{f}(w, \tau) = \frac{\bar{w}(0)}{\bar{w}(\tau)} \hat{f}(v, \tau), \qquad v = \frac{\bar{w}(0)}{\bar{w}(\tau)} w, \tag{59}$$

we have the diffusion equation

$$\frac{\partial}{\partial \tau} \hat{f}(v, \tau) = \frac{B(\tau)}{2} \frac{\partial^2}{\partial v^2} (v^2 \hat{f}(v, \tau)).$$

This yields the asymptotic lognormal behavior

$$\hat{f}(v, \tau) = \frac{1}{v(2\log(\bar{E}(\tau)/\bar{w}(0)^2)\pi)^{1/2}} \exp\left(-\frac{(\log(v) + \log(\sqrt{\bar{E}(\tau)}/\bar{w}(0)))^2}{2\log(\bar{E}(\tau)/\bar{w}(0)^2)}\right), \quad (60)$$

with

$$\int_0^\infty \hat{f}(v, \tau)v \, dv = \bar{w}(0), \qquad \bar{E}(\tau) = \int_0^\infty \hat{f}(v, \tau)v^2 \, dv.$$

6 Numerical Examples

In this section we report the results of different numerical simulations of the proposed kinetic equations. In all the numerical tests, we use $N = 1000$ agents and $n = 10000$ shares. Initially, each investor has a total wealth of 1000 composed of 10 shares, at a value of 50 per share, and 500 in bonds. The random variables ξ and η are assumed distributed according to truncated normal distributions so that negative wealth values are avoided (no borrowing and no short selling). In Tests 1 and 2 we compare the results obtained with the Monte Carlo simulation of the kinetic model to a direct solution of the price equation (35). In the last test case we consider the time-averaged Monte Carlo asymptotic behavior of the kinetic model and compare its numerical self-similar solution with the explicit one computed in the last section using the Fokker-Planck model.
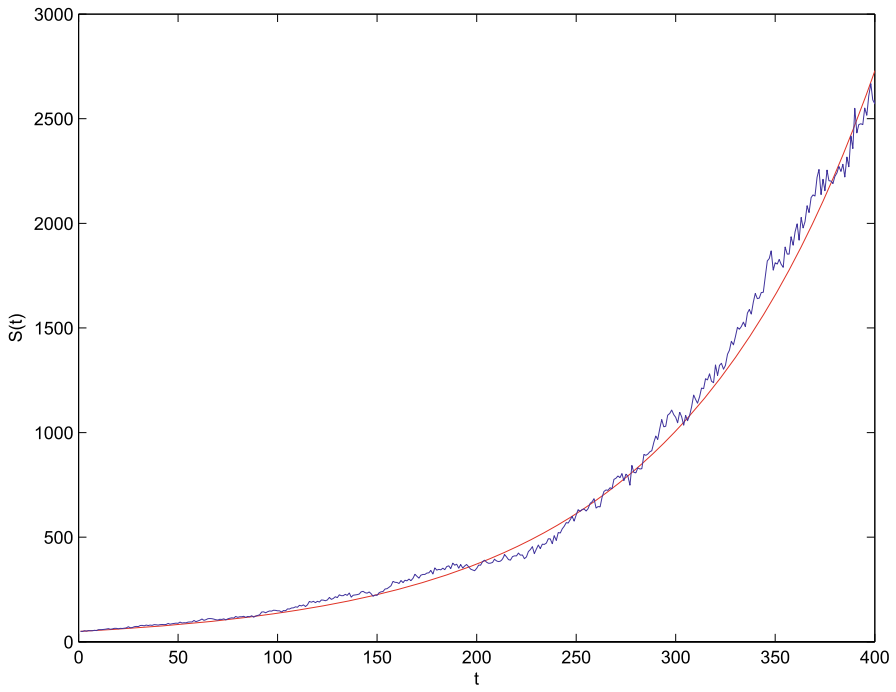
Test 1

In the first test we take a riskless interest rate $r = 0.01$ and an average dividend growth rate $D = 0.015$ and assume that the agents simply follow a constant investments rule, $\mu(\cdot) = C$, with $C \in (0, 1)$ constant. As a consequence of our choice of parameters we have $C = 0.5$ and the evolution of the mean wealth and of the price in the kinetic model are known explicitly (38). We report the results after 400 stock market iterations with ξ and $\eta/S(0)$ distributed with standard deviation 0.2 and 0.3 respectively. In Fig. 2 we report the simulated price behavior together with the evolution computed from (35). The fraction of investments in time during the Monte Carlo simulation fluctuates around its optimal value and is given in Fig. 3.
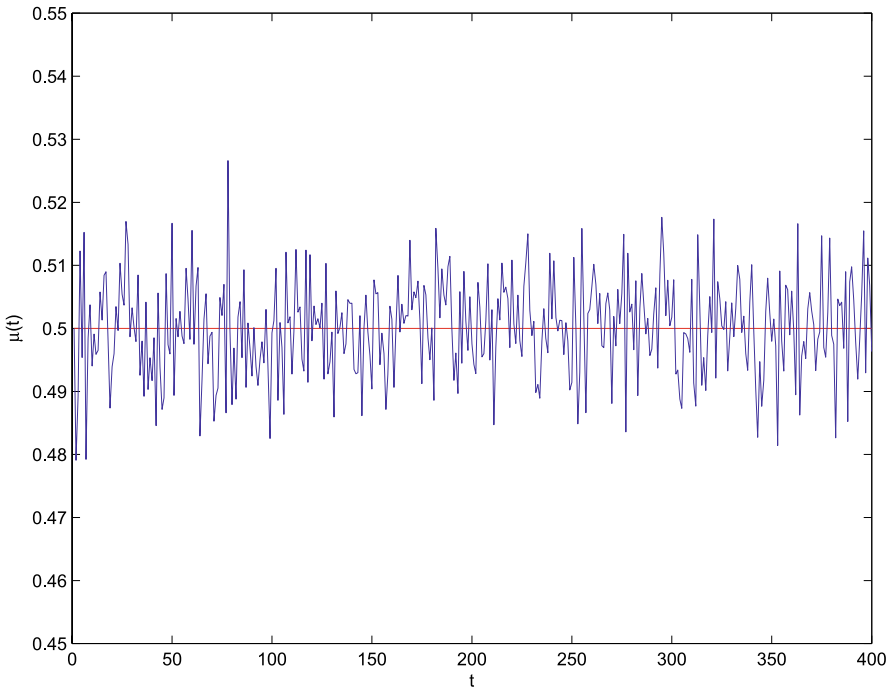
Test 2

In the next test case we take the same parameters as in Test 1 but with a non constant profile $\mu(\cdot)$. More precisely we take a monotone decreasing exponential law

$$\mu(S) = C_1 + (1 - C_1)e^{-C_2 S}$$

with $C_1 = 0.2$ and $C_2 = \log((1 - C_1)/(0.5 - C_1))/S_0 \approx 0.02$ so that the price equation is satisfied for $S_0 = 50$. We have $0.2 < \mu(\cdot) \leq 0.5$. The results for the price evolution and the investments behavior are plotted in Figs. 4 and 5. The solution for the price in the kinetic equation has been computed by direct numerical discretization of (35). Note that the final
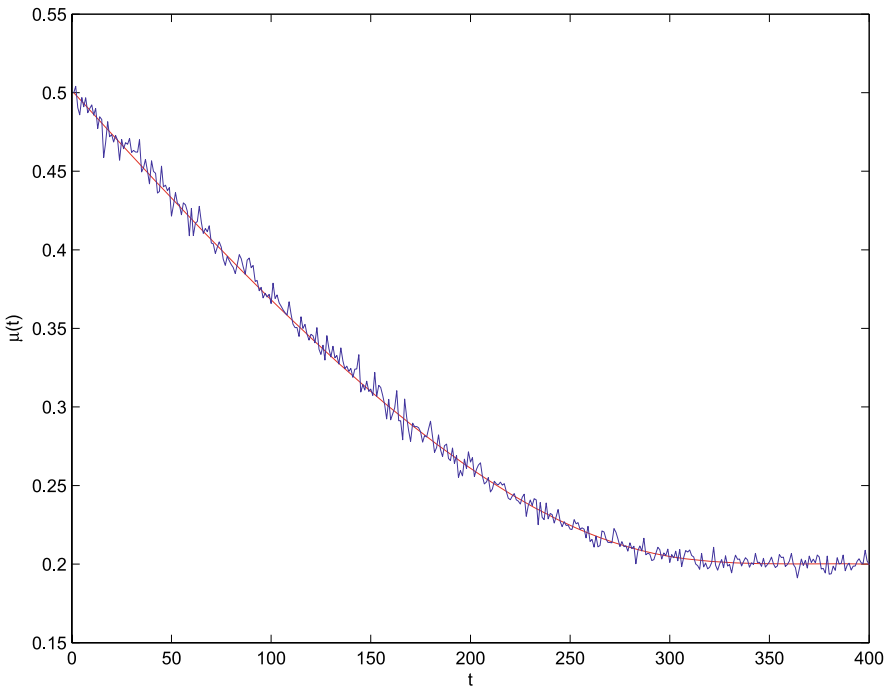
Fig. 2 Test 1. Exponential growth of the price in time. Numerical simulation of the kinetic model
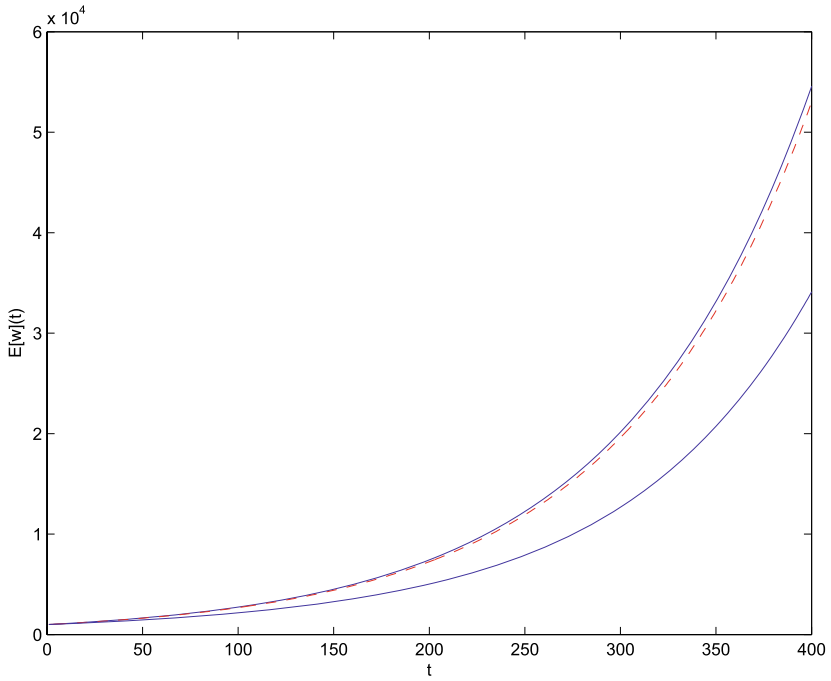

Fig. 3 Test 1. Fluctuations of the corresponding fraction of investments in time
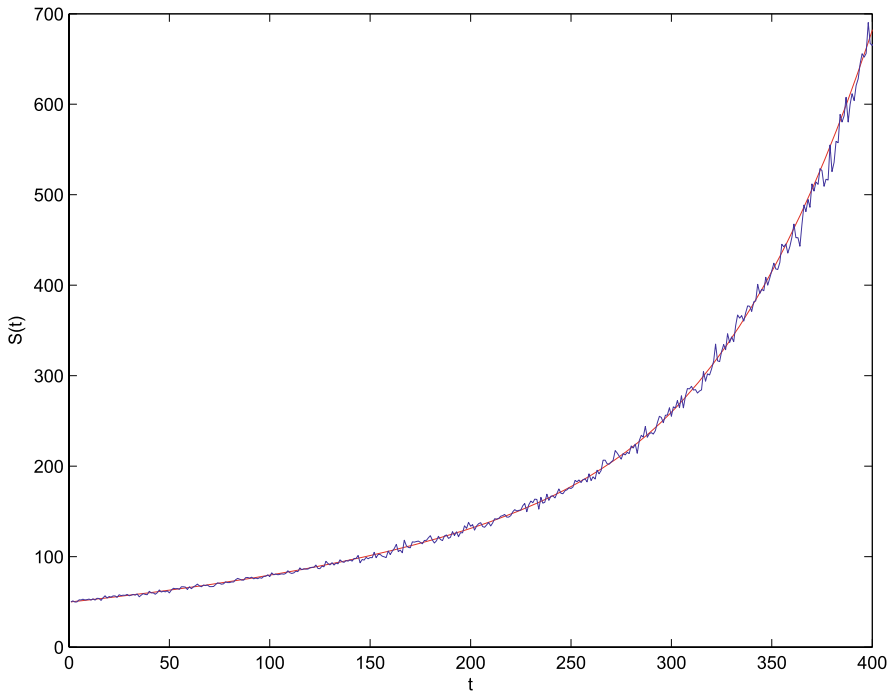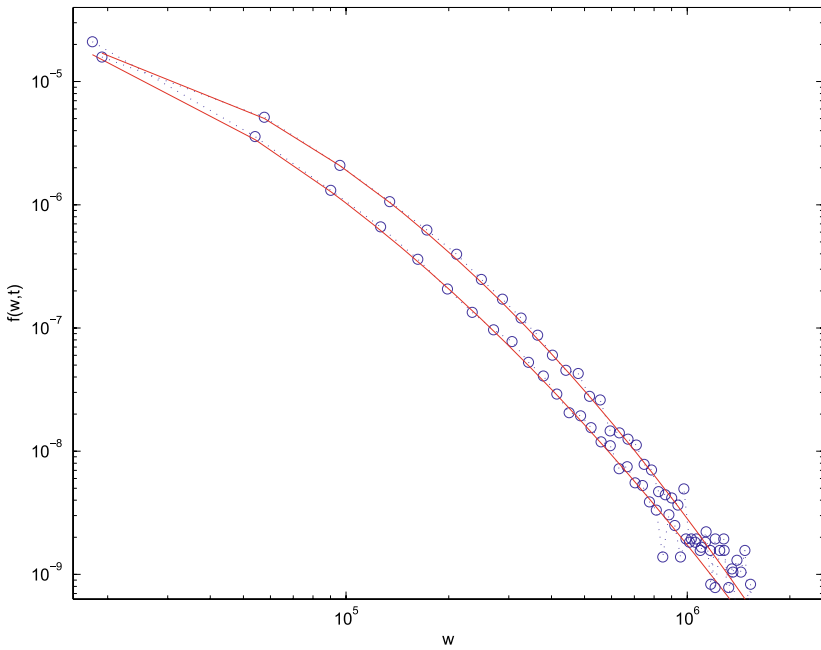
Fig. 4 Test 2. Price behavior in time. Numerical simulation of the kinetic model


Fig. 5 Test 2. Fluctuations of the corresponding fraction of investments in time

Fig. 6 Tests 1 and 2. Behavior of the mean wealth in the kinetic model. The *top curve* refers to Test 1, the *bottom curve* to Test 2. The *dashed line* corresponds to the exponential growth at a rate equal to r

Fig. 7 Tests 1 and 2. Log-log plot of the mean wealth distribution together with a lognormal fitting. The *top curve* refers to Test 1, the *bottom curve* to Test 2
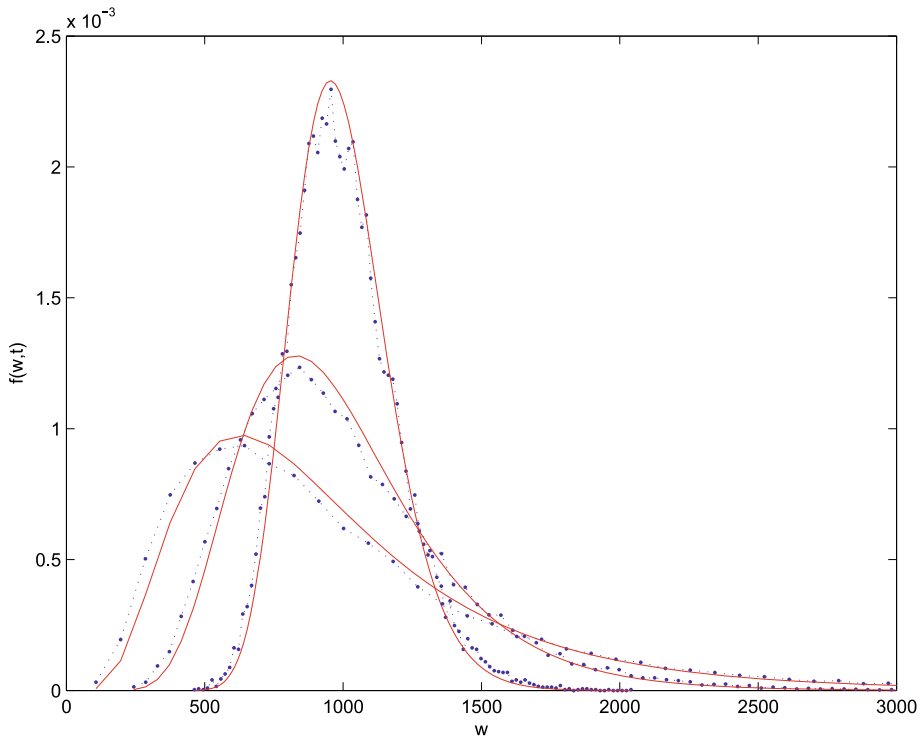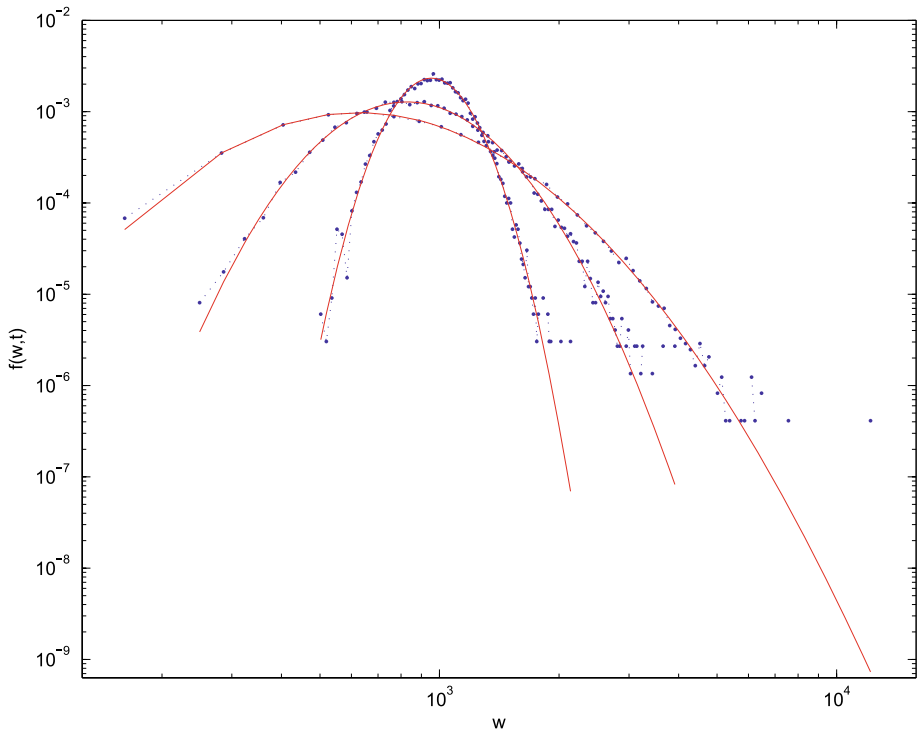
Fig. 8 Test 3. Distribution function at $t = 50, 200, 500$. The *continuous line* is the lognormal Fokker-Planck solution

price is approximatively 5 times smaller then the one in the constant investment case with $\mu = 0.5$. In Fig. 6, we compare the behavior of the mean wealth in Tests 1 and 2 to the exponential growth at a rate r obtained with simple investments in bonds. We can observe that the time decay of investments in Test 2 is fast enough to produce a wealth growth below the rate r. On the contrary, as observed in Sect. 3, a constant investment strategy produces a curve above this rate. Finally, in Fig. 7, we plot the averaged wealth distributions at the final computation time on a log-log scale together with a lognormal fit. The results show lognormal behavior of the tails even for the Boltzmann model. Note that thanks to (3), the same distribution is observed for the number of stocks owned by the agents.

Test 3

In the last test case we consider the asymptotic limit of the Boltzmann model and compare its numerical self-similar solution with the explicit one computed in the last section using the Fokker-Planck model. To this end, we consider the self-similar scaling (59) and compute the solution for the values $r = 0.001$, $D = 0.0015$ with ξ and $\eta/S(0)$ distributed with standard deviation 0.05. We report the numerical solution for a constant value of $\mu = 0.5$ at different times $t = 50, 200, 500$ in Figs. 8 and 9. A very good agreement between the Boltzmann and the lognormal Fokker-Planck solutions is observed, as expected from the results of the last

Fig. 9 Test 3. Log-log plot of the distribution function at $t = 50, 200, 500$. The *continuous line* is the log-normal Fokker-Planck solution

section. We also compute the corresponding Lorentz curve $L(F(w, t))$ defined as

$$L(F(w, t)) = \frac{\int_0^w f(v, t)v\,dv}{\int_0^\infty f(v, t)v\,dv}, \quad F(w, t) = \int_0^w f(v, t)\,dv,$$

and the Gini coefficient $G \in [0, 1]$

$$G = 1 - 2\int_0^1 L(F(w, t))\,dw.$$

The Gini coefficient is a measure of the inequality in the wealth distribution [10]. A value of 0 corresponds to the line of perfect equality depicted in Fig. 10 together with the different Lorentz curves. It is clear that inequalities grow in time due to the speculative dynamics.

7 Conclusions

We have derived a simple linear mesoscopic model which describes a financial market under the assumption that the distribution of investments is known as a function of the price. The model is able to describe the exponential growth of the price of the stock and the growth of the wealth above the rate produced by simple investments in bonds. The long-time behavior
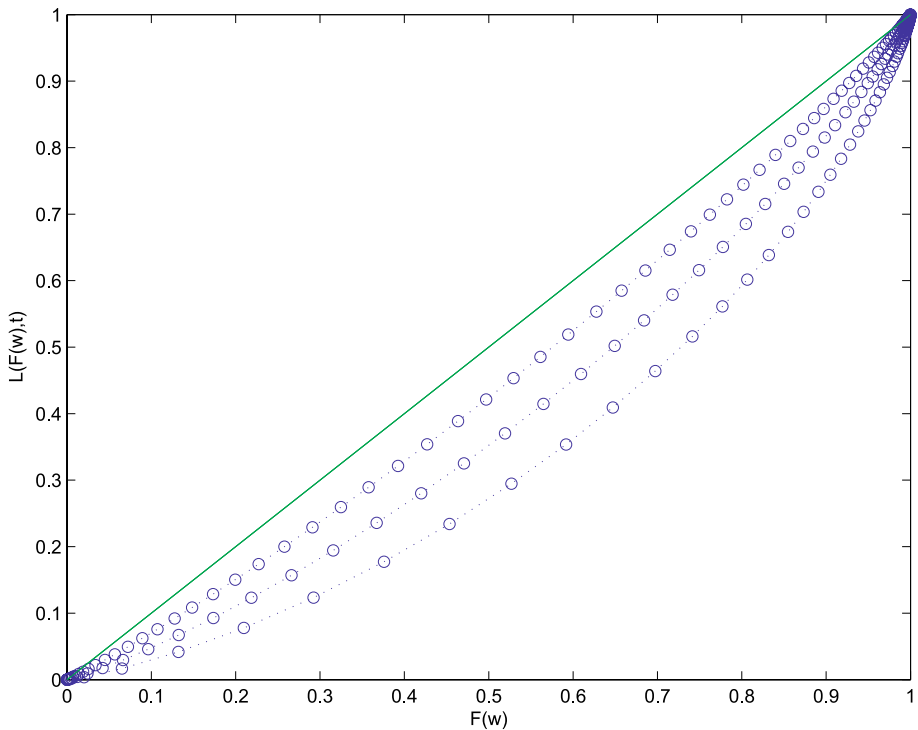
Fig. 10 Test 3. The corresponding Lorentz curves. The Gini coefficients are $G = 0.1$, $G = 0.2$ and $G = 0.3$ respectively

of the model has been studied with the help of a Fokker-Planck approximation. The emergence of a power law tail for the wealth distribution of lognormal type has been proved. In order to produce the effect of a real financial market, with booms, cycles and crashes, the distribution of investments should be a function of time (the decision-making should be done by maximizing the expected utility) and one should consider heterogeneous populations of investors as in [15, 16]. In this case, the model should be modified and the time evolution of $\mu(S, t)$ considered. Another interesting research direction is related to the possibility to introduce stock options into the model and to relate the kinetic approach to Black-Scholes type equations. All these subjects are actually under investigation and we hope to present other challenging results in the near future.

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